Week 7

2020A Adv. Cal. II

Another Fubini's theorem

Let $B = [a,b]\times[c,d]\times[e,f]$ and f defined in B.

Recall

$$\iiint_B f \approx \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

$$= \sum_{i,j}\left(\sum_k f(x_i^*, y_j^*, z_k^*)\,\Delta z_k\right)\Delta x_i\,\Delta y_j$$

$$\approx \sum_{i,j} \int_e^f f(x_i^*, y_j^*, z)\,dz\ \Delta x_i\,\Delta y_j$$

$$\approx \iint_{[a,b]\times[c,d]} \int_e^f f(x,y,z)\,dz\,dA(x,y)$$

$$= \int_a^b\int_c^d\int_e^f f(x,y,z)\,dz\,dy\,dx,$$ which

is the formula we have been using. However, we can put the bracket in a different way.

$$\iiint_B f \approx \sum f(x_i^*, y_j^*, z_k^*)\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

$$= \sum_k\left(\sum_{i,j} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i\,\Delta y_j\right)\Delta z_k$$

$$\approx \sum_k \iint_{[a,b]\times[c,d]} f(x,y,z_k^*)\,dA(x,y)\,\Delta z_k$$

$$= \int_e^f \iint_{[a,b]\times[c,d]} f(x,y,z)\,dA(x,y)\,dz.$$

When Ω is bounded between two planes $z=e$, $z=f$, let

$$\Omega_z = \{ (x,y) : (x,y,z) \in \Omega \}$$

be the cross section of Ω at height z. For $\Omega \subset B$,

$$\iiint_\Omega f = \iiint_B \tilde{f}$$

$$= \int_e^f \iint_{[a,b]\times[c,d]} \tilde{f}(x,y,z)\, dA(x,y)\, dz$$

$$= \int_e^f \iint_{\Omega_z} f(x,y,z)\, dA(x,y)\, dz \qquad (\because \tilde{f}(x,y,z)=0 \text{ for } (x,y) \notin \Omega_z)$$

— ①

In particular, taking $f \equiv 1$ in Ω, we get

$$\text{vol}\, \Omega = \iiint_\Omega 1\, dV$$

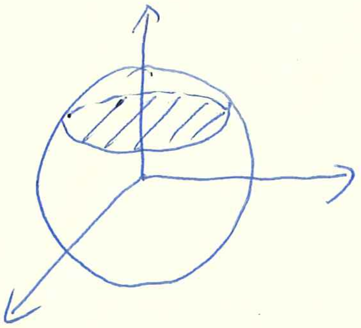
$$= \int_e^f \iint_{\Omega_z} 1\, dA(x,y)\, dz$$

$$= \int_e^f |\Omega_z|\, dz, \quad |\Omega_z| \text{ — area of } \Omega_z .$$

— ②

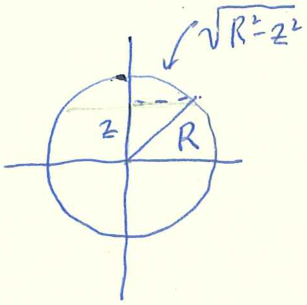
Formulas ① and ② are another forms of Fubini's theorem.

e.g. Find the volume of the ball $x^2+y^2+z^2 \le R^2$ using ②



The cross section at height z is a disk of radius $\sqrt{R^2-z^2}$, as seen from

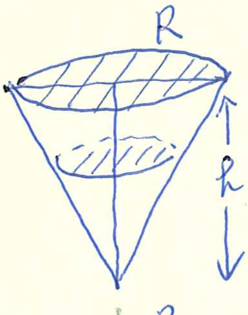
$$x^2+y^2+z^2 \le R^2$$
$$x^2+y^2 \le R^2-z^2$$



So
$$|\Omega_z| = \pi\left(\sqrt{R^2-z^2}\right)^2 = \pi(R^2-z^2).$$

$$\text{Vol of the ball} = 2\int_0^R \pi(R^2-z^2)\,dz = 2\pi\left(R^2z-\frac{z^3}{3}\right)\Big|_0^R = \frac{4\pi}{3}R^3 \quad \#$$

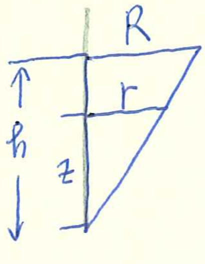
e.g. Find the volume of the circular cone $z=\frac{h}{R}\sqrt{x^2+y^2}$.



$$\frac{r}{z}=\frac{R}{h} \Rightarrow r=\frac{R}{h}z,$$

the cross section at z is a disk of radius $\frac{R}{h}z$.

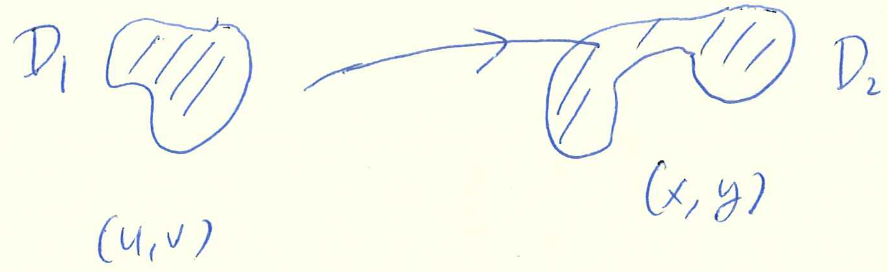
$$\therefore |\Omega_z| = \pi\left(\frac{R}{h}z\right)^2$$



$$\text{Vol} = \int_0^h \pi\left(\frac{R}{h}z\right)^2 dz = \frac{R^2}{h^2}\pi\frac{h^3}{3} = \frac{1}{3}\pi R^2 h \quad \#$$

Change of Variables Formula

Let $x = g(u,v)$, $y = h(u,v)$ be a change of variables



Assumptions

1. $(u,v) \mapsto (x,y)$ maps D_1 onto D_2
2. the interior of D_1 is mapped 1-1 to the interior of D_2
3. g and h are continuously differentiable.
4. the inverse map from the interior of D_2 to the interior of D_1 is also continuously diff.

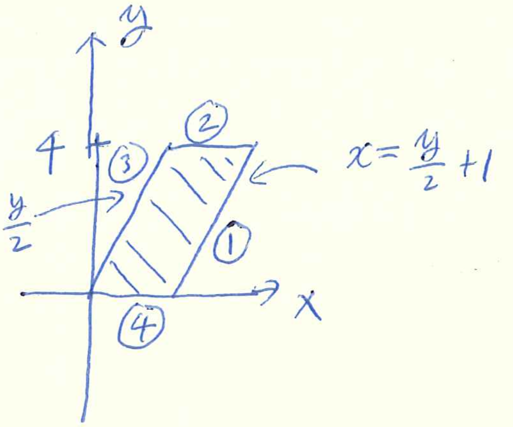
Under ①–④,

$$\iint_{D_2} f = \iint_{D_1} \hat{f}(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v).$$

Here $\hat{f}(u,v) = f(g(u,v), h(u,v))$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = g_u h_v - g_v h_u .$$

e.g. $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy.$

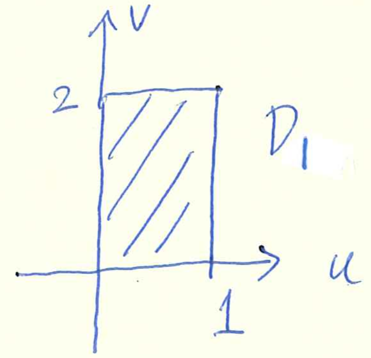


Here D_2 is $\quad \frac{y}{2} \le x \le \frac{y}{2}+1$

$\qquad 0 \le y \le 4$

Introduce $u = \dfrac{2x-y}{2}$, $V = \dfrac{y}{2}$,

To describe D_1, we see how the sides 1 – 4 correspond:



1 $\; x = \frac{y}{2}+1, \quad x - \frac{y}{2} = 1, \quad u = 1$

2 $\; y = 4, \quad V = 2$

3 $\; x = \frac{y}{2}, \quad x - \frac{y}{2} = 0, \quad u = 0$

4 $\; y = 0, \; V = 0$

So D_1 is bounded by $u=1, V=2, u=0, V=0$

Now we calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$u = \frac{2x-y}{2}, \quad V = \frac{y}{2}$$

$$\Rightarrow u+v = x, \quad y = 2V$$

$\therefore \begin{cases} x = u+v \\ y = 2v \end{cases}$ is the change of variables

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 > 0$$

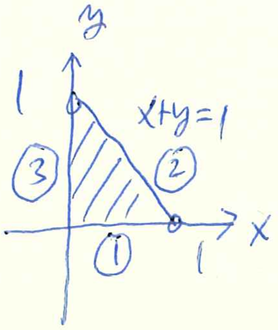
Finally,

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \iint_{D_2} \frac{2x-y}{2}\,dA(x,y)$$

$$= \iint_{D_2} u \, 2 \, dA(u,v)$$

$$= \int_0^1 \int_0^2 2u \, dv \, du$$

$$= 2 \quad \#$$



D_2

e.g. $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy \, dx$

To simplify the integrand, try

$u = x+y, \; v = y-2x.$

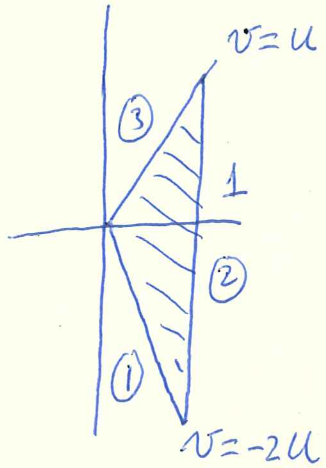
① $y=0$, then $u=x, v=-2x$

$\therefore v=-2u$

② $x+y=1$, $u=1$

③ $x=0$, $u=y, v=y$

$\therefore u=v$



D_1

Next, solve $u=x+y, v=y-2x$ to get

$$\begin{cases} x = \dfrac{u}{3} - \dfrac{v}{3} \\ y = \dfrac{2u}{3} + \dfrac{v}{3} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Finally, $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy \, dx = \iint_{D_2} \sqrt{x+y}\,(y-2x)^2 \, dA(x,y)$

$$= \iint_{D_1} \sqrt{u}\, v^2 \frac{1}{3}\, dA(u,v)$$

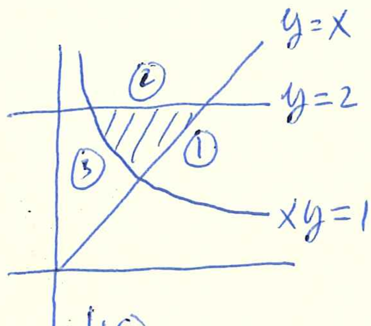
$$= \frac{1}{3} \int_0^1 \int_{-2u}^{u} u^{\frac{1}{2}} v^2 \, dv\, du$$

$\vdots$

$$= \frac{2}{9} \ \#$$

e.g. $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

Let $u = \sqrt{xy}$, $v = \sqrt{y/x}$ simplify the integrand.



D_2 : $1/y \le x \le y$

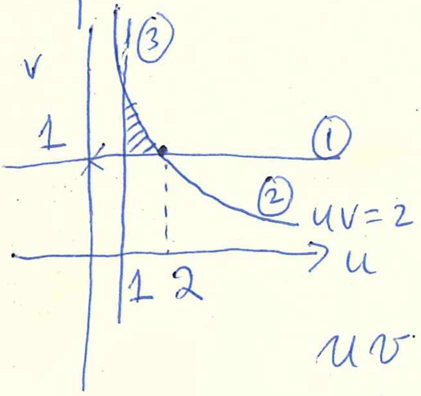
$1 \le y \le 2$

① $y = x$, $u = \sqrt{xy} = x$

$v = 1$

② $y = 2$, $u = \sqrt{2x}$, $v = \sqrt{\frac{2}{x}}$, $uv = 2$

③ $xy = 1$, $u = 1$



$$uv = \sqrt{xy}\sqrt{\frac{y}{x}} = y$$

$$\frac{u}{v} = \frac{\sqrt{xy}}{\sqrt{y/x}} = x$$

$$\therefore \begin{cases} x = u/v \\ y = uv \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$

$$\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy = \int_1^2\int_1^{\frac{2}{u}} v e^{u}\,\frac{2u}{v}\,dv\,du$$

$$= \int_1^2\int_1^{\frac{2}{u}} 2ue^{u}\,dv\,du$$

$$\vdots$$

$$= 2e(e-2) \ \#$$

e.g. Let D_2 be described as

$$g_1(\theta) \le r \le g_2(\theta)$$
$$\theta_1 \le \theta \le \theta_2$$

that is, already in polar form, D_1.

$$\iint_{D_2} f(x,y)\,dA(x,y) = \iint_{D_1} f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dA(r,\theta).$$

As $x = r\cos\theta$, $y = r\sin\theta$,

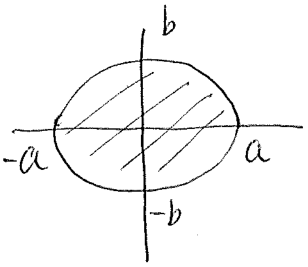
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

So, $$\iint_{D_2} f(x,y)\,dA(x,y) = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta,$$

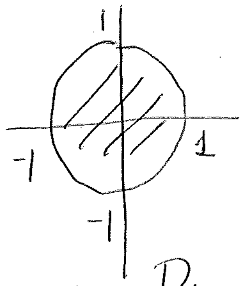
the old formula.

e.g. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $u=\frac{x}{a}$, $v=\frac{y}{b}$, ie $x=au$, $y=bu$.



D_2



D_1.

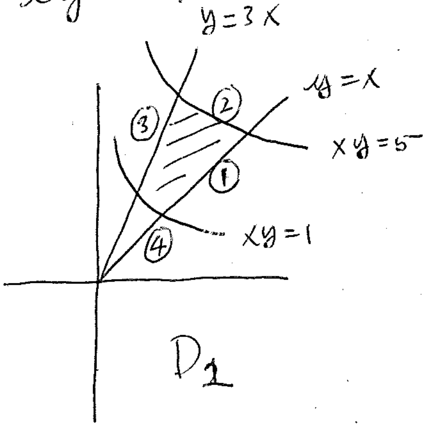
$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}=ab>0$$

D_2 : $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

D_1 : $u^2+v^2\le 1$

$$\text{area}=\iint_{D_2} dA(x,y)=\iint_{D_1} ab\,dA(u,v)=ab\iint_{D_1} dA(u,v)=ab\pi \quad \#$$

e.g. Find the area bdd by the curves $y=x$, $y=3x$, $xy=1$, $xy=5$.

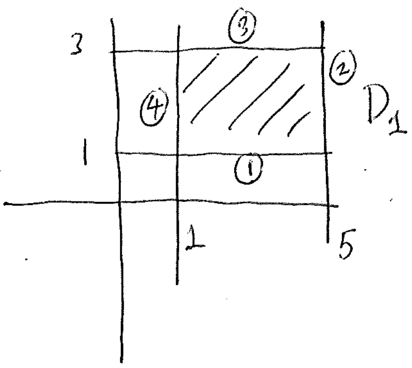


Set $u=xy$, $v=\frac{y}{x}$

Then $uv=y^2 \Rightarrow y=\sqrt{uv}=u^{\frac12}v^{\frac12}$

$\frac{u}{v}=\frac{x\cancel{y}}{\frac{\cancel{y}}{x}} \Rightarrow x=\sqrt{\frac{u}{v}}=u^{\frac12}v^{-\frac12}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac12 u^{-\frac12}v^{-\frac12} & -\frac12 u^{\frac12}v^{-\frac32} \\ \frac12 u^{-\frac12}v^{\frac12} & \frac12 u^{\frac12}v^{-\frac12} \end{vmatrix}=\frac14 v^{-1}-\left(-\frac14\right)v^{-1}=\frac12 v^{-1}.$$



$$\text{area}=\iint_{D_2} 1\,dA(x,y)$$
$$=\iint_{D_1}\frac12 v^{-1}\,dA(u,v)$$
$$=\int_1^5\int_1^3 \frac12 v^{-1}\,dv\,du=\frac12\ln 3\int_1^5 du$$
$$=2\ln 3 \quad \#$$